

# CYCLIC HOMOLOGY OF THE TAFT ALGEBRAS AND OF THEIR AUSLANDER ALGEBRAS

Rachel Taillefer\*

## Abstract

In this paper, we compute the cyclic homology of the Taft algebras and of their Auslander algebras. Given a Hopf algebra  $\Lambda$ , the Grothendieck groups of projective  $\Lambda$ -modules and of all  $\Lambda$ -modules are endowed with a ring structure, which in the case of the Taft algebras is commutative ([C2], [G]). We also describe the first Chern character for these algebras.

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## 1 Introduction

The object of this paper is to compute the cyclic homology and the Chern characters of the Taft algebras  $\Lambda_n$  and of their Auslander algebras  $\Gamma_{\Lambda_n}$ , in order to study a possible influence of the Hopf algebra structure of  $\Lambda_n$  on them.

Note that Auslander algebras are useful when considering artin algebras of finite representation type, since there is a bijection between the Morita equivalence classes of such algebras and the Morita equivalence classes of Auslander algebras (cf [ARS]).

The Hopf algebra structure on an algebra  $\Lambda$  conveys an additional structure on the Grothendieck groups  $K_0(\Lambda)$  and  $\overline{K}_0(\Lambda)$  of isomorphism classes of projective (respectively all) indecomposable modules, since the tensor product over the base ring  $k$  of two  $\Lambda$ -modules is again a  $\Lambda$ -module, via the comultiplication of  $\Lambda$ . Furthermore, there is a one-to-one correspondance between the indecomposable modules over any algebra and the indecomposable projective modules over its Auslander algebra; in the case of a Hopf algebra, therefore, the Grothendieck group of projective modules of  $\Gamma_{\Lambda}$  is endowed with a multiplicative structure. However, this correspondance does not preserve the underlying vector spaces, and this multiplicative structure doesn't seem natural.

In this paper, we study the example of the Taft algebras; they are Hopf algebras which are neither commutative, nor cocommutative. They are interesting for various reasons; for instance,  $\Lambda_p$  is an

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\*Laboratoire G.T.A., Département de Mathématiques CC 51, Université Montpellier II, 34095 Montpellier Cedex 5. email: taillefr@math.univ-montp2.fr

example of a non-semisimple Hopf algebra whose dimension is the square of a prime (cf [M1]). They are of finite representation type; furthermore, when  $n$  is odd,  $\Lambda_n$  is isomorphic to the half-quantum group  $u_q^+(\mathfrak{sl}_2)(q \text{ primitive } n^{\text{th}}\text{-root of unity})$ , and is the only half-quantum group  $u_q^+(\mathfrak{g})$  at a root of unity which is not of wild representation type (cf [C1]). Then, for each  $n$ ,  $\Lambda_n$  is not braided, but its Grothendieck group is a commutative ring nonetheless (cf [C2], [G]).

These examples show that the non-commutative, non-cocommutative Hopf algebra structure of  $\Lambda_n$  does not yield a natural multiplicative structure on its cyclic homology. There is a product, however, obtained by transporting that of  $K_0(\Lambda)$  via the Chern characters, which are onto.

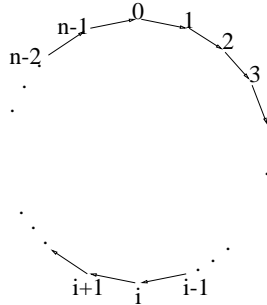
The paper is organized as follows: first, we recall the Hochschild homology of truncated quiver algebras, among which are the Taft algebras, established by Skldberg in [S]. Then, we compute the cyclic homology of these algebras, using the fact that they are graded. In the next section, we study the Auslander algebras of the Taft algebras: their quivers (which are the Auslander-Reiten quivers of the  $\Lambda_n$ ), their Hochschild homology, and their cyclic homology. Finally, we compute the Chern characters of the  $\Lambda_n$  and of the  $\Gamma_{\Lambda_n}$ .

## 2 The Taft algebras

### 2.1 Hochschild homology of truncated quiver algebras

In this paragraph,  $k$  is any commutative ring, except when a root of unity is needed.

Let  $\Delta_n$  be the following quiver (the  $n$ -crown): it has  $n$  vertices  $e_0, \dots, e_{n-1}$ , and  $n$  edges  $a_0, \dots, a_{n-1}$ , each edge  $a_i$  going from the vertex  $e_i$  to the vertex  $e_{i+1}$  for  $0 \leq i \leq n-2$ , and the edge  $a_{n-1}$  going from  $e_{n-1}$  to  $e_0$ , as follows:



Let  $\mathfrak{m}$  be the ideal in the path algebra of  $\Delta_n$  generated by the paths of length 1.

**Definition 2.1** *The Taft algebra  $\Lambda_n$  is the quotient of the path algebra  $k\Delta_n$  by the ideal  $\mathfrak{m}^n$ . When  $k$  contains a primitive  $n^{\text{th}}$  root of unity  $\zeta$ , it can also be described with generators and relations as follows:  $\Lambda_n$  is the algebra generated by two elements  $g$  and  $x$ , subject to the relations  $g^n = 1$ ,  $x^n = 0$ , and  $xg = \zeta gx$ .*

Still in the case where  $k$  contains a primitive  $n^{\text{th}}$  root of unity  $\zeta$ , the algebra  $\Lambda_n = k\Delta/\mathfrak{m}^n$  is a Hopf algebra (see [C1]), with the following structure maps:

$$\begin{aligned} \varepsilon(e_i) &= \delta_{i,0}, & \varepsilon(a_i) &= 0, \\ \Delta(e_i) &= \sum_{j+k=i} e_j \otimes e_k, & \Delta(a_i) &= \sum_{j+k=i} (e_j \otimes a_k + q^k a_j \otimes e_k), \\ S(e_i) &= e_{-i}, & S(a_i) &= -q^{i+1} a_{-i-1}, \end{aligned}$$

where  $\delta$  is the Kronecker symbol.

In [S], for any quiver  $\Gamma$ , Skldberg computes the Hochschild homology of the algebra  $A := k\Gamma/\mathfrak{m}^n$  with coefficients in itself, when  $n \geq 2$ ; to state this result, we shall need some notation: let  $\mathcal{C}$  denote the set of cycles in the quiver  $\Gamma$ , and for any cycle  $\gamma$  in  $\mathcal{C}$ , let  $L(\gamma)$  denote its length. There is a natural action of the cyclic group  $\langle t_\gamma \rangle$  of order  $L(\gamma)$  on  $\gamma$ ; let  $\bar{\gamma}$  denote the orbit of  $\gamma$  under this action, and let  $\bar{\mathcal{C}}$  denote the set of orbits of cycles.

**Theorem 2.2 ([S])** *Set  $q = cn + e$ , for  $0 \leq e \leq n - 1$  ( $n \geq 2$ ). Then:*

$$HH_{p,q}(A) = \begin{cases} k^{a_q} & \text{if } 1 \leq e \leq n - 1 \text{ and } 2c \leq p \leq 2c + 1, \\ \oplus_{r|q} (k^{(n \wedge r)-1} \oplus \text{Ker}(\cdot \frac{n}{n \wedge r} : k \rightarrow k))^{b_r} & \text{if } e = 0, \text{ and } 0 < 2c = p, \\ \oplus_{r|q} (k^{(n \wedge r)-1} \oplus \text{Coker}(\cdot \frac{n}{n \wedge r} : k \rightarrow k))^{b_r} & \text{if } e = 0, \text{ and } 0 < 2c - 1 = p, \\ k^{\#\Gamma_0} & \text{if } p = q = 0, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where  $a_q$  is the number of cycles of length  $q$  in  $\bar{\mathcal{C}}$ ,  $b_r$  is the number of cycles of length  $r$  in  $\bar{\mathcal{C}}$  which are not powers of smaller cycles, and  $n \wedge r$  is the greatest common divisor of  $n$  and  $r$ .

**Example:** The Hochschild homology of the Taft algebra  $\Lambda_n$  is given by:

$$\begin{aligned} HH_{p,cn}(\Lambda_n) &= k^{n-1} \text{ if } p = 2c \text{ or } p = 2c - 1, \\ HH_{0,0}(\Lambda_n) &= k^n \\ HH_{p,q}(\Lambda_n) &= 0 \text{ in all other cases.} \end{aligned}$$

We shall now look at the cases  $n = 0$  and  $n = 1$ . The case  $n = 1$  is quite simple:  $k\Gamma/\mathfrak{m}$  is equal to  $k\Gamma_0 \cong \times_{s \in \Gamma_0} ks$ , so that

$$HH_p(k\Gamma/\mathfrak{m}) = \bigoplus_{s \in \Gamma_0} HH_p(ks) = \begin{cases} \bigoplus_{s \in \Gamma_0} ks & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$$

Finally, for the case  $n = 0$ , we can state:

**Proposition 2.3** *The Hochschild homology of  $k\Gamma$  is given by:*

$$\begin{cases} HH_0(k\Gamma) = k\bar{\mathcal{C}}, \\ HH_1(k\Gamma) = \{\sum_{i=0}^{L(\gamma)-1} t_\gamma^i(\gamma) / \gamma \in \bar{\mathcal{C}}, L(\gamma) \geq 1\} \\ HH_p(k\Gamma) = 0 \text{ if } p \geq 2. \end{cases}$$

**Proof:** We shall use the following resolution (see for instance [C2]):

**Lemma 2.4 ([C2] theorem 2.5)** *There is a  $k\Gamma$ -bimodule projective resolution of  $k\Gamma$  given by*

$$\dots 0 \longrightarrow k\Gamma \otimes_{k\Gamma_0} k\Gamma_1 \otimes_{k\Gamma_0} k\Gamma \longrightarrow k\Gamma \otimes_{k\Gamma_0} k\Gamma \longrightarrow k\Gamma \longrightarrow 0.$$

Tensoring by  $k\Gamma$  over  $k\Gamma^e$  yields the following complex:

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & k\Gamma \otimes_{k\Gamma_0^e} k\Gamma_1 & \xrightarrow{\delta} & k\mathcal{C} & \longrightarrow & 0. \\ & & & & \nu \otimes a & \mapsto & \nu a - a\nu & & \end{array}$$

The space  $HH_0(k\Gamma)$  is generated by the cycles in  $\mathcal{C}$ , subjected to the relations given by the image of  $\delta$ . Since  $\delta(\nu \otimes a) = \nu a - t_{\nu a}(\nu a)$ , the relations identify two cycles in the same orbit, and  $HH_0(k\Gamma) = k\overline{\mathcal{C}}$ .

The complex is  $\overline{\mathcal{C}}$ -graded; therefore, to find  $HH_1(k\Gamma) = \ker \delta$ , it is sufficient to consider elements of type  $x = \sum_{i=0}^{L(\gamma)-1} \lambda_i t_\gamma^i(\gamma)$ , in which  $\nu \otimes a$  is identified with  $\nu a$ , and the  $\lambda_i$  belong to  $k$ . We have  $\delta(x) = 0$  iff  $\lambda_0 = \lambda_1 = \dots = \lambda_{L(\gamma)-1}$ , and the result follows.  $\square$

## 2.2 Cyclic homology of graded algebras

In this paragraph,  $k$  is a commutative ring which contains  $\mathbb{Q}$ . When  $A$  is a graded  $k$ -algebra, Connes' SBI exact sequence splits in the following way:

**Theorem 2.5 ([L] Theorem 4.1.13)** *Let  $A$  be a unital graded algebra over  $k$  containing  $\mathbb{Q}$ . Define  $\overline{HH}_p(A) = HH_p(A)/HH_p(A_0)$  and  $\overline{HC}_p(A) = HC_p(A)/HC_p(A_0)$ . Connes' exact sequence for  $\overline{HC}$  reduces to the short exact sequences:*

$$0 \rightarrow \overline{HC}_{n-1} \rightarrow \overline{HH}_n \rightarrow \overline{HC}_n \rightarrow 0.$$

This will enable us to compute the cyclic homology of truncated quiver algebras. Let us first consider the cases  $n = 0$  and  $n = 1$ . Combining the results for Hochschild homology and theorem 2.5 yields the following:

**Proposition 2.6** *The cyclic homologies of  $k\Gamma$  and of  $k\Gamma/\mathfrak{m}$  are given by:*

$$\begin{aligned} HC_{2c}(k\Gamma/\mathfrak{m}) &= \bigoplus_{s \in \Gamma_0} ks \\ HC_{2c+1}(k\Gamma/\mathfrak{m}) &= 0 \\ &\text{and} \\ HC_0(k\Gamma) &= k\overline{\mathcal{C}} \\ HC_{2c}(k\Gamma) &= k^{\#\Gamma_0} \\ HC_{2c+1}(k\Gamma) &= 0, \end{aligned}$$

for all  $c \in \mathbb{N}$ .

The case  $n \geq 2$  involves the same methods:

**Proposition 2.7** *Suppose  $n \geq 2$ . Then:*

$$\begin{aligned} \dim_k HC_{2c}(k\Gamma/\mathfrak{m}^n) &= \#\Gamma_0 + \sum_{e=1}^{n-1} a_{cn+e} - \sum_{\substack{r|(c+1)n \\ n \notin r\mathbb{N}}} (r \wedge n - 1)b_r \\ \dim_k HC_{2c+1}(k\Gamma/\mathfrak{m}^n) &= \sum_{r|n} (r - 1)b_r. \end{aligned}$$

**Proof:** In the first place,  $A_0$  is equal to  $k\Gamma_0$ , so that we know the homologies of  $A_0$  (see proposition 2.6). Next, we have  $HC_0(A) = HH_0(A) = k^{\#\Gamma_0 + \sum_{e=1}^{n-1} a_e}$ . Then, using theorem 2.5, we get the following formula:

$$\dim_k HC_{2c}(A) + \dim_k HC_{2c+1}(A) = \#\Gamma_0 + \sum_{r|(c+1)n} (r \wedge n - 1)b_r + \sum_{e=1}^{n-1} a_{cn+e}.$$

In particular,  $\dim_k HC_1(A) = \sum_{r|n} (r - 1)b_r$ .

An induction on  $c$  yields the result.  $\square$

**Corollary 2.8** *When  $\Gamma = \Delta_n$  is the  $n$ -crown, then the results are:*

$$\begin{cases} HC_{2c}(\Lambda_n) = k^n, \\ HC_{2c+1}(\Lambda_n) = k^{n-1} \text{ for } c \in \mathbb{N}. \end{cases}$$

**Remark 2.9** *The cyclic cohomology of these truncated quiver algebras has been computed in [BLM] and [Li].*

### 3 The Auslander algebra of $\Lambda_n$

In this section,  $k$  is an algebraically closed field.

#### 3.1 The quiver of the Auslander algebra

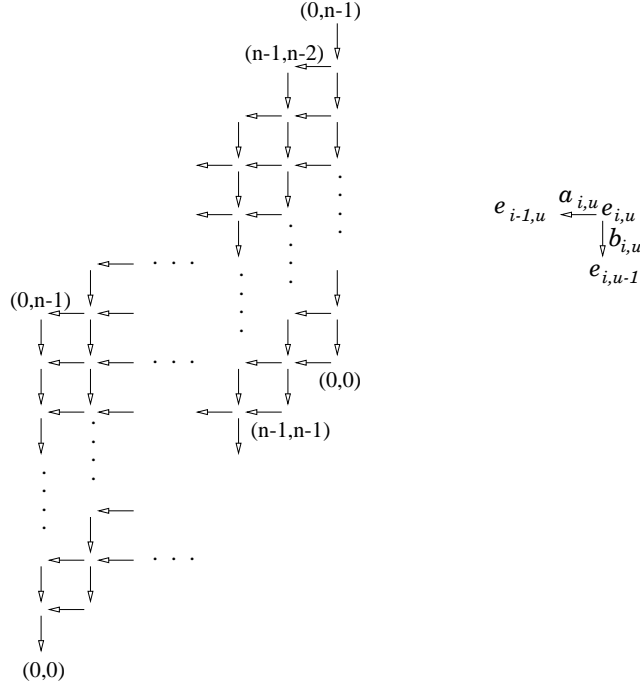
**Definition 3.1** *Let  $\Lambda$  be a finite-dimensional basic algebra over an algebraically closed field  $k$ , with only a finite number of isomorphism classes of indecomposable modules. The Auslander algebra of  $\Lambda$  is:*

$$\Gamma_\Lambda := \text{End}_\Lambda(\oplus_{M \in \text{ind} M} M)^{op},$$

where  $\text{ind} M$  is the set of isomorphism classes of indecomposable  $\Lambda$ -modules.

The algebra  $\Gamma_\Lambda$  has a quiver, which is the opposite of the Auslander-Reiten quiver of  $\Lambda$ . The relations on this quiver are given by the mesh relations (see [ARS] p232).

When the algebra  $\Lambda$  is  $\Lambda_n$ , with  $n \geq 1$ , the quiver is the following:



where both vertical outer edges are identified (the quiver is on a cylinder: see [GR]). Let  $Q$  denote this quiver, let  $\{e_{i,u}/(i,u) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}\}$  be the set of vertices of  $Q$ , and let  $\{a_{i,u}; b_{i,u}/(i,u) \in \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}\}$  be the set of edges of  $Q$ , as in the figure above.

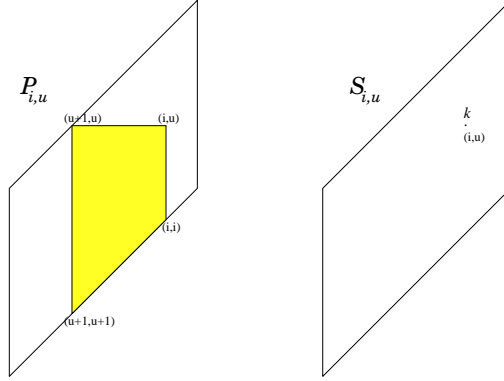
The mesh relations on this quiver are:  $a_{i,i-2}b_{i,i-1} = 0$  for all  $i \in \mathbb{Z}/n\mathbb{Z}$  (the composition of two edges of any ‘triangle’ under the top diagonal is zero), and  $a_{i,iu-1}b_{i,u} + b_{i-1,u}a_{i,u} = 0$  for all  $i$  and  $u$  in  $\mathbb{Z}/n\mathbb{Z}$  (the squares are anticommutative).

The algebra  $\Gamma_{\Lambda_n}$  is the quotient of the path algebra  $kQ$  by the ideal generated by these relations.

**Remark 3.2** *The algebra  $\Gamma_{\Lambda_n}$  is not a Hopf algebra, since its quiver is not a Cayley graph (see [GS] theorem 2.3; in relation to this, see also [CR], in which the authors study the case without relations). Another argument can be given: if  $\Gamma_{\Lambda_n}$  were a Hopf algebra, it would be selfinjective as an algebra (cf [M2]), therefore of homological dimension 0 or  $\infty$ ; however, the homological dimension of an Auslander algebra is at most 2 in general (see [ARS]), and in this case it is not zero, since  $\Lambda_n$  itself is not semisimple ([ARS] proposition 5.2 p211).*

### 3.2 Projective resolutions of simple $\Gamma_{\Lambda_n}$ –modules

Let  $P_{i,u}$  denote the indecomposable projective  $\Gamma_{\Lambda_n}$ –module at the vertex  $e_{i,u}$ , and let  $S_{i,u} = \text{top}(P_{i,u})$  be the corresponding simple module. These modules are described on the quiver by:



where each vertex in the shaded part is represented by  $k$ , and each edge in the shaded part is represented by  $id$ . Everywhere else, the vertices and edges are represented by  $0$  (see also [GR] section 2).

We can compute the minimal projective resolutions of the simple  $\Gamma_{\Lambda_n}$ -modules; the results are as follows:

**Proposition 3.3** *The projective resolutions of the simple modules which are obtained by successive projective covers are:*

$$\begin{array}{ccccccc}
& 0 & \longrightarrow & P_{i-1,i} & \longrightarrow & P_{i,i} & \longrightarrow S_{i,i} \longrightarrow 0 \\
0 \longrightarrow & P_{i-1,i-2} & \longrightarrow & P_{i,i-2} & \longrightarrow & P_{i,i-1} & \longrightarrow S_{i,i-1} \longrightarrow 0 \\
0 \longrightarrow & P_{i-1,i-j-1} & \longrightarrow & P_{i-1,i-j} \oplus P_{i,i-j-1} & \longrightarrow & P_{i,i-j} & \longrightarrow S_{i,i-j} \longrightarrow 0
\end{array}$$

for  $2 \leq j \leq n-1$ .

**Proof:** We consider only the  $S_{n-1,u}$ , because the other cases may be obtained by translating the quiver along the cylinder on which it lies. We then determine the radical of  $P_{n-1,u}$ , and the latter's projective cover, through their representations on the quiver. Iterating this process until the radical obtained is projective yields the result.

In particular, this enables us to compute the  $Ext_{\Gamma_{\Lambda_n}}^p(S; T)$ , where  $S$  and  $T$  are two simple modules:

**Proposition 3.4** *Let  $S$  be a simple  $\Gamma_{\Lambda_n}$ -module. Then:*

$$\begin{aligned}
Ext_{\Gamma_{\Lambda_n}}^1(S_{i,u}; S) &= \begin{cases} k & \text{if } S = S_{i,u}, \\ 0 & \text{if } S \neq S_{i,u}, \end{cases} \\
Ext_{\Gamma_{\Lambda_n}}^2(S_{i,i}; S) &= 0 \\
Ext_{\Gamma_{\Lambda_n}}^2(S_{i,i-j}; S) &= \begin{cases} k & \text{if } S = S_{i-1,i-j-1}, \\ 0 & \text{if } S \neq S_{i-1,i-j-1}, \end{cases} \text{ for } 1 \leq j \leq n-1 \\
Ext_{\Gamma_{\Lambda_n}}^p(S_{i,u}; S) &= 0 \text{ for } p \geq 3.
\end{aligned}$$

**Proof:** The  $Ext^1$  are known, owing to Schur's lemma. The other  $Ext^i$  are obtained from the resolutions of proposition 3.3, applying the functor  $Hom_{\Gamma_{\Lambda_n}}(-; S)$ .

### 3.3 Hochschild and cyclic homologies of $\Gamma_{\Lambda_n}$

We are going to use the previous results to compute a minimal projective resolution of  $\Gamma_{\Lambda_n}$  as a  $\Gamma_{\Lambda_n}$ -bimodule, due to Happel (he does it in the general situation in [H] 1.5.):

**Theorem 3.5 ([H])** *If*

$$\dots \rightarrow R_p \rightarrow R_{p-1} \rightarrow \dots \rightarrow R_1 \rightarrow R_0 \rightarrow \Gamma_{\Lambda_n} \rightarrow 0$$

*is a minimal projective resolution of  $\Gamma_{\Lambda_n}$  as a  $\Gamma_{\Lambda_n}$ -bimodule, then*

$$R_p = \bigoplus_{\substack{(i,u) \\ (j,v)}} (\Gamma_{\Lambda_n} e_{j,v} \otimes e_{i,u} \Gamma_{\Lambda_n})^{\dim_k \text{Ext}_{\Gamma_{\Lambda_n}}^p(S_{i,u}; S_{j,v})}.$$

*Specifically:*

$$\begin{aligned} R_0 &= \bigoplus_{(i,u)} \Gamma_{\Lambda_n} e_{i,u} \otimes e_{i,u} \Gamma_{\Lambda_n} \\ R_1 &= \bigoplus_{(i,u)} [(\Gamma_{\Lambda_n} e_{i-1,u} \otimes e_{i,u} \Gamma_{\Lambda_n}) \oplus (\Gamma_{\Lambda_n} e_{i,u-1} \otimes e_{i,u} \Gamma_{\Lambda_n})] \\ R_2 &= \bigoplus_{\substack{(i,u) \\ i \neq u}} \Gamma_{\Lambda_n} e_{i-1,u-1} \otimes e_{i,u} \Gamma_{\Lambda_n} \\ R_p &= 0 \text{ if } p \geq 3. \end{aligned}$$

Applying the functor  $\Gamma_{\Lambda_n} \otimes_{\Gamma_{\Lambda_n} - \Gamma_{\Lambda_n}} -$ , we obtain a complex:

$$\dots 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow kQ_0 \longrightarrow 0.$$

This yields:

**Proposition 3.6** *The Hochschild homology of  $\Gamma_{\Lambda_n}$  is:*

$$\begin{cases} HH_0(\Gamma_{\Lambda_n}) = kQ_0 \cong k^{n^2} \\ HH_p(\Gamma_{\Lambda_n}) = 0 \end{cases} \quad \forall p \in \mathbb{N}^*,$$

*and hence the cyclic homology of  $\Gamma_{\Lambda_n}$  is:*

$$\begin{cases} HC_{2p}(\Gamma_{\Lambda_n}) = kQ_0 \cong k^{n^2} \\ HH_{2p+1}(\Gamma_{\Lambda_n}) = 0 \end{cases} \quad \forall p \in \mathbb{N}.$$

**Remark 3.7** *There doesn't seem to be any connection between these results and those for  $\Lambda_n$  (see the example on page 3 and corollary 2.8).*



## 4 Chern characters of $\Lambda_n$ and $\Gamma_{\Lambda_n}$

Let  $K_0(\Lambda_n)$  (resp.  $K_0(\Gamma_{\Lambda_n})$ ) be the Grothendieck group of projective  $\Lambda_n$ -modules (resp.  $\Gamma_{\Lambda_n}$ -modules). We are interested in the Chern characters  $ch_{0,p} : K_0(\Lambda_n) \rightarrow HC_{2p}(\Lambda_n)$  (resp.  $K_0(\Gamma_{\Lambda_n}) \rightarrow HC_{2p}(\Gamma_{\Lambda_n})$ ). We shall write  $[P_j]$  (resp.  $[P_{i,u}]$ ) for the isomorphism class of the projective module at the vertex  $e_j$  (resp.  $e_{i,u}$ ).

Set  $\sigma^p = (y_p, z_p, \dots, y_1, z_1, y_0) \in \mathbb{N}^{2p+1}$  with  $y_p = (-1)^p(2p)!/p!$  and  $z_p = (-1)^{p-1}(2p)!/2(p!)$ . There is a system of generators of  $HC_{2p}(\Lambda_n)$  (resp.  $HC_{2p}(\Gamma_{\Lambda_n})$ ) given by the following set:

$$\{\sigma_i^p := \sigma^p(e_i, \dots, e_i) \in (TotCC(\Lambda_n))_{2p} / i = 0, \dots, n-1\}$$

(resp. by  $\{\sigma_{i,u}^p := \sigma^p(e_{i,u}, \dots, e_{i,u}) \in (TotCC(\Gamma_{\Lambda_n}))_{2p} / i, u \in \{0, 1, \dots, n-1\}\}$ ).

Consider the elements

$$\begin{aligned} \epsilon_j : \Lambda_n &\longrightarrow \Lambda_n \\ \lambda &\longmapsto \lambda e_j \end{aligned}$$

in  $\mathcal{M}_1(\Lambda_n)$  and

$$\begin{aligned} \epsilon_{i,u} : \Gamma_{\Lambda_n} &\longrightarrow \Gamma_{\Lambda_n} \\ \lambda &\longmapsto \lambda e_{i,u} \end{aligned}$$

in  $\mathcal{M}_1(\Gamma_{\Lambda_n})$ ; their ranges are the corresponding projective modules. Then by definition of the Chern characters (see [L] 8.3.4), we have:

$$\begin{aligned} ch_{0,p}([P_j]) &= ch_{0,p}([\epsilon_j]) := \text{tr}(c(\epsilon_j)) = \sigma_j^p \text{ in } HC_{2p}(\Lambda_n) \\ ch_{0,p}([P_{i,u}]) &= ch_{0,p}([\epsilon_{i,u}]) = \sigma_{i,u}^p. \end{aligned}$$

using the isomorphisms  $\mathcal{M}_m(\Lambda) \cong \mathcal{M}_m(k) \otimes \Lambda$ . Here,

$$c(\epsilon_j) = (y_p \epsilon_j^{\otimes 2p+1}, z_p \epsilon_j^{\otimes 2p}, \dots, z_1 \epsilon_j^{\otimes 2}, y_0 \epsilon_j) \in \mathcal{M}(\Gamma_{\Lambda_n})^{\otimes 2p+1} \oplus \dots \oplus \mathcal{M}(\Gamma_{\Lambda_n}).$$

**Remark 4.1** *There is a decomposition formula for the tensor product of indecomposable modules on  $\Lambda_n$  (see [C2], [G]). From this formula, we get inductively:*

$$ch_{0,p}([L_1] \otimes \dots \otimes [L_r]) = (1/n^2) \prod_{i=1}^r (\dim L_i) (\sigma_0^p, \dots, \sigma_{n-1}^p), \text{ for } r \geq 2,$$

where the  $L_i$  are arbitrary projective  $\Lambda_n$ -modules. Unfortunately, this product in the cyclic homology doesn't seem natural.

**Remark 4.2** *Let  $\overline{K}_0(\Lambda_n)$  be the Grothendieck group of all  $\Lambda_n$ -modules (not just the projective ones). Then  $\overline{K}_0(\Lambda_n) \cong K_0(\Gamma_{\Lambda_n})$ . Hence, if  $N_{i,u}$  is the indecomposable  $\Lambda_n$ -module which starts at the vertex  $i$  and ends at the vertex  $u$ , it corresponds to the projective  $\Gamma_{\Lambda_n}$ -module  $P_{i,u}$ , and we get a map:*

$$\begin{aligned} \overline{K}_0(\Lambda_n) &\longrightarrow HC_{2p}(\Gamma_{\Lambda_n}) \\ N_{i,u} &\longmapsto \sigma_{i,u}^p. \end{aligned}$$

**Remark 4.3** Although  $\Gamma_{\Lambda_n}$  is not a Hopf algebra, its Grothendieck group  $K_0(\Gamma_{\Lambda_n})$  does have a ring structure, albeit not natural: for every  $[P]$  in  $K_0(\Gamma_{\Lambda_n})$ , there exists a  $[B]$  in  $K_0(\Lambda_n)$  such that  $[P] = [\text{Hom}_{\Lambda_n}(M, B)]$ , where  $M$  is the sum of all isomorphism classes of indecomposable  $\Lambda_n$ -modules. If  $[Q] = [\text{Hom}_{\Lambda_n}(M, C)]$  is another element in  $K_0(\Gamma_{\Lambda_n})$ , we can set

$$[P].[Q] = [\text{Hom}_{\Lambda_n}(M, B \otimes_k C)]$$

(the vector space  $B \otimes_k C$  is a  $\Lambda_n$ -module since  $\Lambda_n$  is a Hopf algebra). In fact, using the decomposition in [C2] and [G], the product can be written:

$$[P_{i,u}][P_{j,v}] = \begin{cases} \sum_{l=0}^{v-j} [P_{i+j+l, u+v-l}] & \text{if } u+v-(i+j) \leq n-1 \\ \sum_{l=0}^e [P_{i+j+l, u+v+l-1}] + \sum_{m=e+1}^{v-j} [P_{i+j+m, u+v-m}] & \text{if } e := u+v-(i+j)-(n-1) \geq 0. \end{cases}$$

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